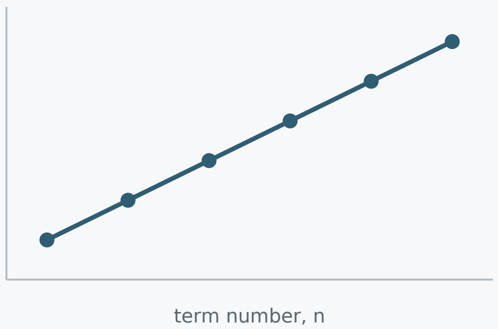
EXAM PREPARATION NOTES

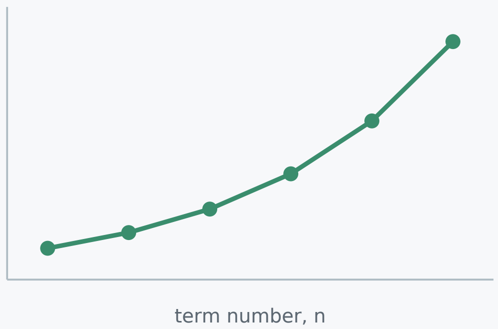
Arithmetic and Geometric Sequences and Series

IB Mathematics: Analysis and Approaches • A Level Mathematics

Arithmetic: equal differences



Geometric: equal ratios



Coverage	Designed for
Terms, sums, sigma notation and recurrence relations	IB AA SL/HL
Finite and infinite geometric series; convergence	A Level Pure Mathematics
Growth, decay, finance, modelling and proof	Class tests and final revision

A concise, exam-focused guide with worked methods, common errors, mixed practice and answers.

01 Revision roadmap

Use the notes actively: diagnose, practise, check, then repeat.

> **The 30-second classification test**
> Ask two questions: **Is there a constant difference?** Then the sequence is arithmetic. **Is there a constant non-zero ratio?** Then it is geometric. If neither test works, look for another pattern or a recurrence.

If the question asks for...	First move	Main tool
A term or term number	Identify u_1 and d or r	nth-term formula
A finite total	Count the terms carefully	S_n formula
An infinite sum	Check $\|r\| < 1$ first	$S_\infty = u_1/(1-r)$
A threshold or minimum n	Write an inequality	Logs, graph or table
A recurrence limit	Set $u_{n+1} = u_n = L$	Fixed-point equation
A context model	Define period and initial value	AP/GP or recurrence

A high-scoring revision cycle

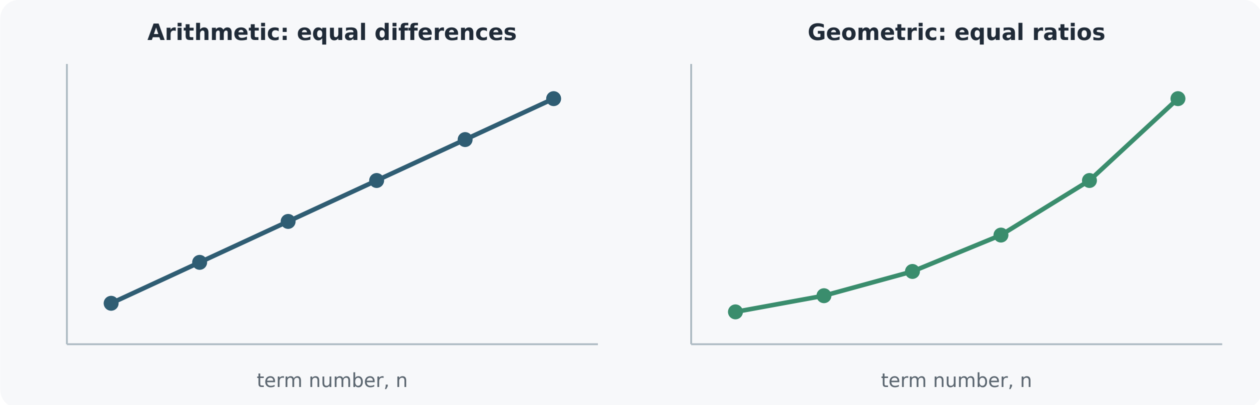
1. **Recall:** write the relevant formula before looking at the worked examples.
2. **Apply:** complete one routine question without notes.
3. **Explain:** state why the model is arithmetic or geometric.
4. **Check:** substitute your result into the original condition and test its reasonableness.
5. **Extend:** attempt a threshold, modelling or proof question.

> **IB/A Level command terms**
> **Find/Calculate** requires working and a value. **Determine** expects a justified result. **Show that** means the given answer must emerge from a clear derivation. **Hence** requires you to use the preceding result.

02 Foundations and notation

Sequences are ordered lists; series are sums of their terms.

Symbol	Meaning	Example
u_n	the nth term	u_4 is the fourth term
u_1	the first term	the starting value
S_n	sum of the first n terms	$S_4 = u_1+u_2+u_3+u_4$
d	common difference	$u_{n+1}-u_n$
r	common ratio	u_{n+1}/u_n
Σ	summation	$\Sigma_{r=1}^{n} u_r$



Explicit and recursive definitions

Type	What it tells you	Example
Explicit	Find u_n directly from n	$u_n = 5 + 3(n-1)$
Recursive	Find the next term from earlier term(s)	$u_{n+1} = u_n + 3$, $u_1 = 5$

Index check
In $u_n = u_1r^{n-1}$ and $u_n = u_1+(n-1)d$, the exponent or multiplier is **n-1** because no change has occurred at the first term.

Useful standard sums

$$\Sigma_{r=1}^{n} 1 = n \qquad \Sigma_{r=1}^{n} r = n(n+1)/2 \qquad \Sigma_{r=1}^{n} r^2 = n(n+1)(2n+1)/6$$

03 Arithmetic sequences

A constant additive change produces a linear nth term.

$$u_n = u_1 + (n-1)d$$

Recognising an arithmetic sequence

- Compute consecutive differences. If every difference is the same, the sequence is arithmetic.
- A negative d gives a decreasing sequence; d = 0 gives a constant sequence.
- The graph of u_n against n consists of discrete points on a straight line with gradient d.

Finding unknown parameters from two terms

If u_p and u_q are known, subtract their equations. The first term cancels:

$$u_q - u_p = (q-p)d$$

Worked example. Given $u_5 = 14$ and $u_{12} = 42$, find u_1, d and u_{30}.

$u_{12}-u_5 = 7d$, so $42-14 = 7d$ and $d = 4$. Then $14 = u_1+4(4)$, giving $u_1 = -2$. Therefore $u_{30} = -2+29(4) =$ **114**.

Three consecutive arithmetic terms

Represent three consecutive terms as **a-d, a, a+d**. Their sum is 3a, so the middle term is their arithmetic mean. This symmetric form is efficient in sum/product questions.

Exam check
When d is negative, keep brackets around it: $u_n = u_1+(n-1)(-4)$. This prevents sign errors.

04 Arithmetic series and sigma notation

Pairing the first and last terms gives the finite-sum formula.

$$S_n = n/2\,[2u_1+(n-1)d] = n/2\,(u_1+u_n)$$

Choose the form that matches the information

Known information	Best form
u_1, d and n	$S_n = n/2\,[2u_1+(n-1)d]$
first term, last term and n	$S_n = n/2\,(u_1+u_n)$
a formula for S_n	$u_n = S_n - S_{n-1}$

Counting terms

For a sequence from u_p to u_q inclusive, the number of terms is **q-p+1**. For a list with known first and last values, solve u_n = last; do not count the gaps by eye.

Worked example. Find the sum 7+11+15+… when the total is 525.
Here $u_n = 7+4(n-1) = 4n+3$. Then $S_n = n/2[7+(4n+3)] = n(2n+5)$. Solve $n(2n+5)=525$: $2n^2+5n-525=0$, so $n=15$ (reject the negative root). The last term is 63.

Linear expressions inside sigma notation

$$\Sigma_{r=1}^{n}(ar+b) = a\Sigma_{r=1}^{n}r + b\Sigma_{r=1}^{n}1 = an(n+1)/2 + bn$$

Common trap
In $\Sigma_{r=p}^{q} f(r)$, the number of terms is q-p+1, not q. Substitute the lower limit to identify the actual first term.

05 Arithmetic methods in exam questions

Translate the wording into term equations before doing algebra.

Example A - sum from two non-adjacent terms

An arithmetic sequence has $u_3 = 8$ and $u_9 = 32$. Find S_{20}.

Subtract: $u_9 - u_3 = 6d = 24$, so $d=4$. Then $u_1+2d=8$, giving $u_1=0$. Therefore $S_{20} = 20/2[2(0)+19(4)] =$ **760**.

Example B - obtaining terms from S_n

Suppose $S_n = 3n^2-2n$. Find u_n and show that the sequence is arithmetic.

$u_n = S_n - S_{n-1} = (3n^2-2n)-[3(n-1)^2-2(n-1)] =$ **6n-5**. This is linear in n, so consecutive terms differ by 6; hence the sequence is arithmetic.

Example C - a real-life linear model

A theatre has 18 seats in the first row and 3 more seats in each successive row. With 24 rows, $u_{24} = 18+23(3)=87$ and $S_{24} = 12(18+87)=$**1260 seats**.

> **Model statement**
> A context is arithmetic only when the **absolute amount** added or removed per equal time interval is constant.

Fast verification

- Check u_1 by substituting $n=1$ into your nth term.
- Check d using u_2-u_1.
- Estimate the sum as n × average term; for an arithmetic series the exact average is $(u_1+u_n)/2$.
- Reject any non-integer or non-positive solution for a number of terms.

06 Geometric sequences

A constant multiplicative change produces an exponential nth term.

$$u_n = u_1 r^{n-1}$$

Recognising a geometric sequence

- Divide each term by the previous term. The ratio must be constant.
- If $r > 1$ and $u_1 > 0$, the terms grow; if $0 < r < 1$, they decay.
- If $r < 0$, signs alternate. The size grows when $|r| > 1$ and shrinks when $|r| < 1$.

Using two non-adjacent terms

$$u_q/u_p = r^{(q-p)}$$

Worked example. Given $u_2 = 15$ and $u_5 = 405$, find u_1, r and u_7.
$u_5/u_2 = r^3 = 405/15 = 27$, so $r=3$. Then $u_1=15/3=5$ and $u_7=5\times3^6=$**3645**.

When there may be two ratios

If the equation gives $r^2 = 9$, both $r=3$ and $r=-3$ must be considered. An odd power such as $r^3=27$ has only the real solution $r=3$.

Percentage multiplier
Growth by p%: $r = 1+p/100$. Decay by p%: $r = 1-p/100$. A decrease of 18% therefore means multiply by 0.82, not by 0.18.

07 Finite geometric series

Multiply by r and subtract to cancel the middle terms.

$$S_n = u_1(1-r^n)/(1-r),\ r \neq 1$$

$$\text{Equivalently, } S_n = u_1(r^n-1)/(r-1).$$

Which version should you use?

Both formulas are identical. The first often feels natural when $|r|<1$; the second avoids double negatives when $r>1$.

Worked example. Find the sum of the first seven terms of 5, 15, 45, …

$u_1=5$, $r=3$, $n=7$. Therefore $S_7 = 5(3^7-1)/(3-1) = 5(2186)/2 =$ **5465**.

Worked example with a negative ratio. Find S_8 for 4, -2, 1, …

Here $r=-1/2$. $S_8 = 4[1-(-1/2)^8]/[1-(-1/2)] = 4(1-1/256)/(3/2) =$ **85/32**.

Sigma notation and the first term

For $\Sigma_{i=1}^{7} 2(1/2)^i$, substitute $i=1$: the first term is 1, not 2. Then $r=1/2$ and $n=7$.

Calculator discipline
Enter the numerator and denominator with full brackets. Keep the exact value until the final line, then round as requested.

08 Infinite geometric series

An infinite sum exists only when the terms approach zero fast enough.

Behaviour of r^n as n increases

$$|r| < 1 \quad \Rightarrow \quad S_\infty = u_1/(1-r)$$

Why the condition matters

The finite formula contains r^n. If $|r|<1$, then $r^n \to 0$ as $n \to \infty$, so $S_n \to u_1/(1-r)$. If $|r| \geq 1$, the terms do not approach zero and the ordinary infinite sum diverges.

Worked example. Find the exact sum 12-4+4/3-4/9+…

The ratio is -1/3, so $|r|<1$. Therefore $S_\infty = 12/[1-(-1/3)] = 12/(4/3) =$ **9**.

Worked example from a tail. A positive geometric series has S_∞=25, and the sum from the third term onward is 9.

The tail-to-whole ratio is r^2, because the third term begins two multiplications after u_1. Thus $r^2=9/25$. Positivity gives $r=3/5$. Then $u_1=25(1-3/5)=10$ and $u_5=10(3/5)^4=$**162/125**.

Necessary first line
Before using S_∞, write **$|r|<1$**. This often earns a method mark and prevents accepting an invalid root.

09 Finding n: equations, inequalities and logs

Threshold questions require the smallest integer that satisfies the original statement.

Terms

$$u_1 r^{n-1} = K \quad \Rightarrow \quad n = 1 + \log(K/u_1)/\log(r)$$

This direct logarithmic form assumes positive quantities and $r>0$, $r\neq1$. For a negative ratio, parity matters and a graph/table is usually safer.

Worked example. An asset is worth 2400 and loses 18% each year. When is it first worth less than 600?

$V_n=2400(0.82)^n$, where n is the number of completed years. Solve $2400(0.82)^n<600$, so $(0.82)^n<0.25$. Because $\log(0.82)<0$, the inequality reverses when dividing: $n>6.985\ldots$. The smallest integer is **$n=7$**; $V_7\approx598.29$.

Sums

Worked example. For $u_1=6$ and $r=0.92$, find the least n for which $S_n>70$.

$S_n=6(1-0.92^n)/0.08 = 75(1-0.92^n)$. Thus $75(1-0.92^n)>70$, giving $0.92^n<1/15$. This gives $n>32.48\ldots$, so the least integer is **33**.

Always verify the boundary
Check both n and n-1 in the original equation or inequality. Rounding a logarithmic answer without this check is a common source of lost marks.

10 Recurrence relations and limits

Many linear recurrences become geometric after subtracting a fixed point.

$$u_{n+1} = au_n+b$$

Step 1 - find the fixed point

If the sequence converges to L, then consecutive terms approach the same value. Set $L=aL+b$, giving $\mathbf{L=b/(1-a)}$ when $a\neq1$.

Step 2 - shift the sequence

Subtract L: $u_{n+1}-L = a(u_n-L)$. Therefore u_n-L is geometric with ratio a.

$$u_n = L + (u_1-L)a^{n-1}$$

Worked example. $u_1=500$ and $u_{n+1}=0.8u_n+60$.

$L=0.8L+60$, so $L=300$. Hence $u_n=300+(500-300)(0.8)^{n-1}$ =$\mathbf{300+200(0.8)^{n-1}}$.

Sum of the first 10 terms

$\Sigma_{n=1}^{10}u_n = 10(300)+200\Sigma_{n=1}^{10}(0.8)^{n-1}$ =$3000+200(1-0.8^{10})/(1-0.8) \approx$ **3892.63**.

Convergence condition
For $u_{n+1}=au_n+b$, the shifted geometric part tends to zero when $|a|<1$. If a is negative, the terms may oscillate around L while converging.

11 Growth, decay and compound interest

Translate each percentage change into a multiplier per stated period.

Future value: $A = P(1+i)^n$
Repeated end-of-period deposits: $F = R[(1+i)^n-1]/i$

Situation	Multiplier
growth of p%	1+p/100
decay of p%	1-p/100
annual rate compounded monthly	1 + annual decimal rate/12
deposit at beginning of each period	ordinary-annuity value × (1+i)

Worked example. Deposit £120 at the end of each month for 12 months at 0.4% per month.

The deposits form $120(1.004)^{11}+\dots+120(1.004)+120$. Thus $F=120[(1.004)^{12}-1]/0.004\approx$ **£1472.11**.

Timing diagram in words

- A deposit made now earns interest for every remaining period.
- An end-of-period final deposit earns no interest before the account is valued.
- Count the powers: the oldest deposit has the largest exponent.

Units first
Make n and i refer to the same period. A monthly rate requires a number of months; a yearly rate requires a number of years.

Model limitations

Real models may fail if rates change, deposits are missed, fees or taxes apply, or the percentage is only an average. State one limitation that is specific to the context.

12 Modelling patterns and fractals

Define the stage number carefully and separate ‘each object’ from ‘total’.

Linear versus exponential change

Feature	Arithmetic model	Geometric model
constant quantity	difference d	ratio r
graph of term against n	discrete straight line	discrete exponential curve
typical context	fixed annual increase	fixed percentage growth
long-run growth	linear	exponential if $\|r\|>1$

Koch-type perimeter

Suppose an equilateral triangle has side 81 cm. At each stage, every segment is replaced by four segments, each one third as long. The number of segments is multiplied by 4 while each length is multiplied by 1/3. Therefore the perimeter is multiplied by 4/3.

$$P_n = 243(4/3)^n \quad \text{for stage 0 perimeter 243 cm}$$

Sierpiński-type area

If each stage retains 3 out of 4 equal sub-triangles, the total retained area is multiplied by 3/4. After n stages, $A_n=A_0(3/4)^n$. The number of visible triangles and the area of each triangle form separate geometric sequences.

> **Stage-number convention**
> Write what stage 0 means. If the initial figure is stage 0, the multiplier is raised to n; if it is term 1, the exponent is n-1.

Choosing between competing models

Compare predicted and observed values, inspect residuals, and consider whether the mechanism suggests a fixed difference or fixed percentage. A model can fit early data and still be unrealistic when extrapolated.

13 Proofs you should be able to reproduce

A short derivation can secure method marks and deepen formula recall.

Arithmetic sum by pairing

Write S_n forward and backward. Each column then sums to u_1+u_n, and there are n columns:

$$S_n = u_1 + (u_1+d) + \dots + u_n$$
$$S_n = u_n + (u_n-d) + \dots + u_1$$
$$\mathbf{2S_n = n(u_1+u_n)}$$

Geometric sum by subtraction

Write S_n, multiply by r, then subtract so all middle terms cancel:

$$S_n = u_1 + u_1r + \dots + u_1r^{n-1}$$
$$rS_n = u_1r + u_1r^2 + \dots + u_1r^n$$
$$\mathbf{(1-r)S_n = u_1(1-r^n)}$$

Proof structure for ‘show that’

1. Start from known definitions or the side that is easier to transform.
2. State the substitution or identity you use.
3. Keep equality signs aligned logically; do not jump over essential algebra.
4. Reach the exact required form and stop.

Do not assume the target
In a ‘show that’ question, avoid writing the required result as your first line. Your argument must derive it from accepted facts.

14 Choosing the method

Most exam questions reduce to a small number of reliable templates.

Question pattern	Set-up
Two AP terms known	$u_p=u_1+(p-1)d$ and $u_q=u_1+(q-1)d$
Two GP terms known	$u_q/u_p=r^{(q-p)}$
Sum formula S_n known	$u_n=S_n-S_{n-1}$
Finite total required	identify u_1, d/r and n
Infinite total required	find r, prove $\lvert r\rvert<1$, then use S_∞
Smallest integer n	solve inequality, then boundary-check
Linear recurrence	find fixed point L and shift by L
Regular deposits	draw powers of $(1+i)$ before summing

Exact forms and rounding

- Keep fractions and surds exact unless a decimal is requested.
- For money, show the unrounded calculator value and give the final answer to the nearest cent/penny unless instructed otherwise.
- For a number of terms or periods, the final answer must normally be an integer with contextual wording.

Reasonableness checks

- An increasing positive GP with $r>1$ should not produce a smaller later term.
- For $0<r<1$, a finite partial sum is less than S_∞.
- A negative ratio produces alternating signs.
- A total should have the correct unit; n and r have no units.

> **Communication mark**
> Define variables in context: 'Let n be the number of completed months.' This removes ambiguity about the exponent and the final interpretation.

15 Common errors and how to prevent them

These are small mistakes with a large mark cost.

Error	Why it fails	Prevention
Using n instead of n-1	the first term is shifted	test n=1
Confusing sequence and series	u_n is not S_n	write the requested symbol
Using S_∞ without $\lvert r\rvert<1$	the series may diverge	check convergence first
Treating 8% decay as r=0.08	92% remains	r=1-0.08=0.92
Missing ± for an even power	one valid GP is lost	solve $r^2=c$ with both signs
Rounding n directly	threshold may be crossed late	test adjacent integers
Miscounting sigma terms	wrong n in sum formula	upper-lower+1
Ignoring deposit timing	every exponent shifts	draw a timeline

No-calculator checklist

- Factor before using a quadratic formula.
- Rewrite powers with a common base where possible.
- Use exact fractions and cancel common factors early.
- Show the formula before substitution.

Calculator checklist

- Use brackets around the full power, numerator and denominator.
- Store intermediate values instead of retyping rounded decimals.
- For a threshold, use a table/graph or logs, then verify the boundary.
- Report only the precision requested.

16 Mixed practice I

Try these without looking back. Suggested time: 25 minutes.

1. An arithmetic sequence has $u_4=11$ and $u_{17}=50$. Find u_1 and d.

2. Find the exact sum 5+9+13+…+101.

3. The sum of the first n terms is $S_n=3n^2-2n$. Find u_n and prove that the terms form an arithmetic sequence.

4. A geometric sequence has $u_3=18$ and $u_6=486$. Find u_1 and r.

5. Find the exact sum of the first eight terms of 4,-2,1,… .

6. Determine whether 12-4+4/3-4/9+… converges. If it does, find its sum.

Marking focus
Award yourself method marks only when the correct formula or equation is visible. A correct calculator answer with no supporting work is not secure exam practice.

Reflection

- Which question type did you identify immediately?
- Where did notation, algebra or calculator entry slow you down?
- Can you explain each rejected root or convergence decision?

17 Mixed practice II

Suggested time: 30 minutes. Include units and interpretation.

7. Determine all x for which $\Sigma_{k=1}\infty(1+x/3)^k$ converges. Give the sum in terms of x.

8. A machine worth £48,000 depreciates by 13% per year. Find its value after six years.

9. $u_1=100$ and $u_{n+1}=0.7u_n+12$. Find an explicit formula for u_n and the limiting value.

10. £120 is deposited at the end of each month for 12 months. Interest is 0.4% per month. Find the account value immediately after the final deposit.

11. An equilateral triangle has side 81 cm. At each stage, each segment is replaced by four segments one third as long. Find the perimeter after five stages.

12. A culture begins with 5 thousand cells and grows by 18% per hour. Find the first whole hour when it exceeds 20 thousand cells.

Modelling focus
For Questions 8, 10, 11 and 12, define the period and state what your exponent counts.

18 Solutions to Mixed practice I

Compare structure as well as final answers.

1. $u_{17}-u_4=13d=39$, so $d=3$. Then $u_1+3d=11$, giving $\mathbf{u_1=2}$.

2. $d=4$ and $n=(101-5)/4+1=25$. Therefore $S_{25}=25(5+101)/2=$**1325**.

3. $u_n=S_n-S_{n-1}=(3n^2-2n)-[3(n-1)^2-2(n-1)]=$**6n-5**. Since $u_{n+1}-u_n=6$, the sequence is arithmetic.

4. $u_6/u_3=r^3=486/18=27$, so $r=3$. Then $u_1=18/3^2=$**2**.

5. $u_1=4$, $r=-1/2$. $S_8=4[1-(-1/2)^8]/[1-(-1/2)]=$**85/32**.

6. $r=-1/3$, so $|r|<1$ and the series converges. $S_\infty=12/[1-(-1/3)]=$**9**.

What full-credit work looks like

- The defining formula is shown before numbers are substituted.
- Index differences are correct: $u_{17}-u_4$ spans 13 common differences.
- The convergence condition is stated explicitly.
- Exact values are retained when requested.

19 Solutions to Mixed practice II

Check interpretation and boundary decisions.

7. $r=1+x/3$. Convergence requires $-1<1+x/3<1$, hence $\mathbf{-6<x<0}$. Since $u_1=r$, $S_\infty=r/(1-r)=\mathbf{-(x+3)/x}$.

8. $V=48000(0.87)^6=$**£20,814.06** (to the nearest penny).

9. $L=0.7L+12$, so $L=40$. Therefore $\mathbf{u_n=40+60(0.7)^{n-1}}$, and the limiting value is **40**.

10. $F=120[(1.004)^{12}-1]/0.004=$**£1472.11**.

11. $P_0=3(81)=243$ and the perimeter ratio is $4/3$. $P_5=243(4/3)^5=$**1024 cm**.

12. $5(1.18)^n>20$ gives $n>\log(4)/\log(1.18)=8.38\ldots$. The first whole hour is **9 hours**; check that hour 8 is still below the target.

Self-assessment
If your final number is correct but your model or inequality is missing, redo the question. The goal is a solution that another reader can follow and verify.

20 One-page formula and exam checklist

Use this page for final recall, not as a substitute for practice.

Topic	Formula / condition
Arithmetic term	$u_n=u_1+(n-1)d$
Arithmetic sum	$S_n=n/2[2u_1+(n-1)d]=n/2(u_1+u_n)$
Geometric term	$u_n=u_1r^{n-1}$
Finite geometric sum	$S_n=u_1(1-r^n)/(1-r)$, $r\neq1$
Infinite geometric sum	$\|r\|<1$ and $S_\infty=u_1/(1-r)$
Terms from sums	$u_n=S_n-S_{n-1}$
Linear recurrence	$L=b/(1-a)$; $u_n=L+(u_1-L)a^{n-1}$
Compound growth	$A=P(1+i)^n$
End-of-period deposits	$F=R[(1+i)^n-1]/i$
Number of sigma terms	upper limit - lower limit + 1

Final exam checklist

1. Classify the pattern and define the variables.
2. Write the formula before substitution.
3. Use n-1 in nth-term formulas and count terms inclusively.
4. Check $|r|<1$ before an infinite sum.
5. Keep exact values until the requested rounding point.
6. For a threshold, test the final integer and the preceding one.
7. Attach units and answer in the context of the question.
8. Use a specific limitation when commenting on a model.

> **Last-minute memory cue**
> **Arithmetic = add; geometric = multiply.** Terms use n-1. Finite sums need n. Infinite sums need $|r|<1$.