Sequences, Series and Sigma Notation

IB / A-Level Exam Preparation Notes

Focus of this notes pack

Recognise patterns in number sequences and write clear general terms.

Use proper mathematical notation with subscripts and superscripts, for example u_r and r^2, rather than caret notation.

Expand and write series using sigma notation with correct upper and lower limits.

Build exam confidence through worked examples, common traps and practice questions.

Key ideas

A sequence is an ordered list of terms.

A series is the sum of the terms of a sequence.

A general term allows a sequence to be described without listing every term.

Sigma notation is a compact way to write a finite or infinite series.

1. Exam overview

This section introduces the language of patterns, sequences and series. In exams, questions often ask you to recognise a rule, write a general term, expand sigma notation, or rewrite a visible series in sigma notation.

Concept	Meaning	Exam skill
Pattern	A repeated or predictable structure in numbers, shapes or operations.	Describe the rule clearly.
Generalization	A rule that works for many cases, not just the examples shown.	Write a formula using r or n.
Sequence	An ordered list of terms.	Find next terms or the general term.
Series	The sum of the terms of a sequence.	Write, expand or evaluate sums.
Sigma notation	A compact notation for repeated addition.	Use limits and a general term correctly.

Notation to know

u_r represents the rth term of a sequence.

$\{u_r\}$ represents the sequence as a whole.

$r \in \mathbb{Z}+$ means r is a positive integer: 1, 2, 3, ...

$\sum_{r=1}^{10} r$ means add r from r = 1 to r = 10.

∞ means the series continues indefinitely.

2. Before you start: algebra skills

The chapter assumes that you can solve equations, simplify surds and manipulate algebraic fractions. These skills are useful because sequence rules often need algebraic simplification.

Skill	Example	Exam reminder
Solve linear equations	$x - 3(x + 5) = 20 - 3x$ gives $x = 35$	Expand brackets before collecting terms.
Simplify surds	Rationalise denominators such as $\sqrt{2} / (1 - \sqrt{2})$.	Multiply top and bottom by the conjugate.
Algebraic fractions	Use a common denominator before combining terms.	Factor denominators where possible.

Why this matters for sequences

A general term is a formula. You may need to simplify it, substitute values into it, or compare it with another expression. Algebra accuracy is therefore essential.

3. Sequences and series

A sequence is a list of numbers written in a defined order and following a rule. Each number in the list is called a term. A sequence may be finite or infinite.

Type	Example	Description
Finite sequence	7, 5, 3, 1, -1, -3	The list stops after a fixed number of terms.
Infinite sequence	7, 5, 3, 1, -1, -3, ...	The ellipsis shows that the pattern continues forever.
Finite series	7 + 5 + 3 + 1 - 1 - 3 = 12	A sum with a fixed number of terms.
Infinite series	1 + 3 + 9 + 27 + 81 + ...	A sum that continues indefinitely.

Sequence versus series

Sequence: 2, 5, 8, 11, ...

Series: 2 + 5 + 8 + 11 + ...

A sequence lists terms. A series adds them.

4. The general term

A general term gives a formula for the rth term of a sequence. Instead of listing many terms, one formula can describe the whole pattern.

Core notation

$$u_r = 3r - 1$$

This gives the sequence 2, 5, 8, 11, ... when r = 1, 2, 3, 4, ...

$$u_1 = 3(1) - 1 = 2$$
$$u_2 = 3(2) - 1 = 5$$

Pattern type	Example sequence	General term
Arithmetic pattern	2, 7, 12, 17, ...	$u_r = 5r - 3$
Product pattern	2, 6, 12, 20, ...	$u_r = r(r + 1)$
Fraction pattern	1/2, 2/3, 3/4, 4/5, ...	$u_r = r / (r + 1)$
Geometric pattern	5, 10, 20, 40, ...	$u_r = 5 \times 2^{r - 1}$
Alternating signs	2, -10, 50, -250	$u_r = 2(-5)^{r - 1}, 1 \le r \le 4$

5. Worked examples: finding general terms

Example 1: arithmetic sequence

Find the next three terms and a general term for 2, 7, 12, 17, ...

Solution

The common difference is +5, so the next three terms are 22, 27, 32.

$$u_r = 2 + (r - 1) \times 5$$

$$u_r = 5r - 3$$

Example 2: product pattern

Find the next three terms and a general term for 2, 6, 12, 20, ...

Solution

Rewrite the terms as 1 x 2, 2 x 3, 3 x 4, 4 x 5, ...

The next terms are 5 x 6 = 30, 6 x 7 = 42 and 7 x 8 = 56.

$$u_r = r(r + 1)$$

Example 3: fraction pattern

Find the next three terms and a general term for 1/2, 2/3, 3/4, 4/5, ...

Solution

The numerator is r and the denominator is r + 1.

The next three terms are 5/6, 6/7 and 7/8.

$$u_r = r / (r + 1)$$

Example 4: geometric sequence

Find the next three terms and a general term for 5, 10, 20, 40, ...

Solution

Each term is multiplied by 2. The next three terms are 80, 160 and 320.

The first term is 5, and the multiplier 2 is used one fewer time than the term number.

$$u_r = 5 \times 2^{r - 1}$$

Example 5: alternating signs

Write 2, -10, 50, -250 using a general term.

Solution

Each term is multiplied by -5.

$$u_r = 2(-5)^{r - 1}$$

Because only four terms are given, state the domain:

$$r \in \mathbb{Z}^+, 1 \le r \le 4$$

Exam tip

Always test your formula by substituting r = 1, r = 2 and r = 3. If it does not reproduce the first few terms, revise the rule.

6. Finding terms from a general term

When a general term is given, substitute r = 1, 2, 3, ... to generate the terms of the sequence.

Example 1

$$u_r = 5r - 2$$
$$u_1 = 5(1) - 2 = 3$$
$$u_2 = 5(2) - 2 = 8$$
$$u_3 = 5(3) - 2 = 13$$

First three terms: 3, 8, 13

Example 2

$$u_r = (-1)^r / r^2$$
$$u_1 = (-1)^1 / 1^2 = -1$$
$$u_2 = (-1)^2 / 2^2 = 1/4$$
$$u_3 = (-1)^3 / 3^2 = -1/9$$

First three terms: -1, 1/4, -1/9

Common trap

Do not confuse the term number r with the term itself. For example, when r = 3, the term is the value of the formula after substitution.

7. Sigma notation

Sigma notation is a compact way of writing a series. The Greek capital letter sigma means sum.

General form

$$\sum_{r\,=\,\text{first value}}^{\text{last value}} \text{general term}$$

The lower limit tells you where to start. The upper limit tells you where to stop. The expression after sigma gives the rth term.

Sigma notation	Expanded form
$\sum_{r=1}^{10} r$	1 + 2 + 3 + ... + 10
$\sum_{r=1}^{20} 5r$	5 + 10 + 15 + ... + 100
$\sum_{r=0}^{4} (2r + 1)$	1 + 3 + 5 + 7 + 9
$\sum_{r=1}^{\infty} (-1)^{r-1}$	1 - 1 + 1 - 1 + ...

Reading sigma notation

$\sum_{r=0}^{4} (2r + 1)$ is read as: the sum of 2r + 1 from r = 0 to r = 4.

8. Worked examples: expanding sigma notation

Example 1

$$\sum_{r=1}^{10} r(r - 1)$$
$$= 1(0) + 2(1) + 3(2) + 4(3) + 5(4) + \ldots$$
$$= 0 + 2 + 6 + 12 + 20 + \ldots$$

Example 2

$$\sum_{r=1}^{\infty} (-1)^r r^2$$
$$= (-1)^1 \times 1^2 + (-1)^2 \times 2^2 + (-1)^3 \times 3^2 + \ldots$$
$$= -1 + 4 - 9 + 16 - 25 + \ldots$$

Example 3

$$\sum_{r=1}^{\infty} (r + 1)/(2r - 1)$$
$$= 2/1 + 3/3 + 4/5 + 5/7 + 6/9 + \ldots$$
$$= 2 + 1 + 4/5 + 5/7 + 6/9 + \ldots$$

Exam method

1. Identify the lower limit.
2. Substitute this value into the general term.
3. Increase r by 1 each time.
4. Stop at the upper limit, or continue with ellipsis if the upper limit is ∞.

9. Worked examples: writing sigma notation

To write a series in sigma notation, first find the general term, then decide the starting and ending values of r.

Example 1: arithmetic finite series

Write 3 + 11 + 19 + 27 + 35 in sigma notation.

Solution

The common difference is 8 and the first term is 3.

$$\text{General term} = 3 + (r - 1) \times 8 = 8r - 5$$

$$\sum_{r=1}^{5} (8r - 5)$$

Example 2: alternating infinite series

Write 1 - 1 + 1 - 1 + 1 - 1 + ... in sigma notation.

Solution

The signs alternate, and the first term is positive.

$$\sum_{r=1}^{\infty} (-1)^{r - 1}$$

Example 3: alternating geometric finite series

Write -6 + 12 - 24 + 48 - 96 + 192 in sigma notation.

Solution

Each term is multiplied by -2. The first term is -6 and there are six terms.

$$\sum_{r=1}^{6} -6(-2)^{r - 1}$$

An equivalent form is also acceptable if it generates the same terms.

10. Pattern bank for fast recognition

Visible pattern	What to check	Possible general term
Constant difference	Add or subtract the same number each time.	$u_r = a + (r - 1)d$
Constant multiplier	Multiply by the same number each time.	$u_r = a\,q^{r-1}$
Alternating signs	Terms switch positive and negative.	Use $(-1)^r$ or $(-1)^{r-1}$
Fractions	Look separately at numerators and denominators.	$u_r = r/(r + 1)$, $r/(2r + 1)$, etc.
Products	Rewrite terms as products.	$u_r = r(r + 1)$
Square pattern	Compare with 1, 4, 9, 16, ...	$u_r = r^2$ or $(r + k)^2$

Choosing the sign pattern

If the first term is positive and signs alternate: $(-1)^{r-1}$.

If the first term is negative and signs alternate: $(-1)^r$.

Example: $u_r = (-1)^r r^2$ gives -1, 4, -9, 16, ...

11. Inquiry example: tiles and generalization

In a square room tiled with square tiles, suppose the total number of tiles along both main diagonals is D. If the square has n tiles on each side, the two diagonals together count the central tile twice. Therefore:

Diagonal tile relationship

$$D = 2n - 1$$

$$n = (D + 1)/2$$

This works when D is odd. If D is even, it cannot be the total number of tiles on the two diagonals of a square grid with one shared centre tile.

Diagonal total D	Side length n	Total tiles
9	(9 + 1)/2 = 5	5 x 5 = 25
13	(13 + 1)/2 = 7	7 x 7 = 49
15	(15 + 1)/2 = 8	8 x 8 = 64
135	(135 + 1)/2 = 68	68 x 68 = 4624

Why algebra is useful

A table gives particular cases. A formula gives every possible case, explains why the pattern works, and allows very large examples to be solved quickly.

12. Inquiry example: Koch snowflake

The Koch snowflake is made by replacing the middle third of each side with two sides of an equilateral triangle. This creates more sides at each iteration.

Perimeter pattern

If the starting shape is an equilateral triangle with side length 1 m:

$$P_0 = 3$$
$$P_1 = 3 \times 4/3$$
$$P_2 = 3 \times (4/3)^2$$
$$P_r = 3 \times (4/3)^r$$

Because the multiplier 4/3 is greater than 1, the perimeter increases without bound as r increases. This leads to the surprising idea that a fractal can have an infinite perimeter while enclosing a finite area.

Exam connection

Fractals connect visual patterns to geometric sequences, limits and proof. This is why early pattern work is important before studying advanced series.

13. Common mistakes and how to avoid them

Mistake	Why it is wrong	Better approach
Using the term value as r	r is the position number, not the term.	Substitute r = 1, 2, 3, ... into the formula.
Forgetting the first term shift	Geometric formulas usually use r - 1 in the exponent.	For first term a and ratio q, use $u_r = aq^{r - 1}$.
Wrong alternating sign	The sequence may start positive or negative.	Test $(-1)^r$ and $(-1)^{r - 1}$ with r = 1.
Incorrect sigma limits	The series may not start at r = 1.	Check the first term by substituting the lower limit.
Writing a sequence instead of a series	A sequence lists; a series adds.	Use commas for sequences and plus signs for series.

Best exam habit

After writing any general term or sigma expression, expand the first three terms to check that it matches the given sequence or series.

14. Practice questions

Try these without looking at the answer key. They are designed to practise recognition, notation and expansion.

1. Write the next three terms and a general term for 3, 4.5, 6, 7.5, ...

2. Write the next three terms and a general term for 17, 14, 11, 8, ...

3. Write the next three terms and a general term for 3, 9, 27, 81, ...

4. Write the first five terms of the sequence $u_r = 2r^2 - 1$.

5. Write the first five terms of the sequence $u_r = (-1)^{r - 1} / r$.

6. Expand $\sum_{r=1}^{4} 2r(1 - r)$.

7. Expand the first five terms of $\sum_{r=0}^{5} ((-1)^r) r^2$.

8. Write 8 + 5 + 2 - 1 - 4 in sigma notation.

9. Write 1 + 9 + 25 + 49 + 81 in sigma notation.

10. Write 3k + 6k + 9k + 12k + 15k in sigma notation.

15. Answer key and marking guidance

Use this section to check both the final answer and the method. In exams, method marks are often awarded for a correct pattern even if the final notation contains a small error.

1. Next terms: 9, 10.5, 12. General term: $u_r = 1.5r + 1.5$.
2. Next terms: 5, 2, -1. General term: $u_r = 20 - 3r$.
3. Next terms: 243, 729, 2187. General term: $u_r = 3^r$.
4. 1, 7, 17, 31, 49.
5. 1, -1/2, 1/3, -1/4, 1/5.
6. $\sum_{r=1}^{4} 2r(1 - r) = 0 - 4 - 12 - 24$.
7. 0, -1, 4, -9, 16.
8. $\sum_{r=1}^{5} (11 - 3r)$.
9. $\sum_{r=1}^{5} (2r - 1)^2$.
10. $\sum_{r=1}^{5} 3kr$.

Skill	What earns marks
Next terms	Correct continuation of the pattern.
General term	A formula that gives all listed terms using r as the position.
Sigma expansion	Correct substitution of each r value and correct signs.
Sigma notation	Correct lower limit, upper limit and general term.
Presentation	Use clear notation with proper subscripts and superscripts.